Homework 1

Lambda Calculus (Master MPRI M1)

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Homework

Write the proof down carefully and submit it before next course

Prove that if $x \neq y$ *and* $x \notin fv(v)$ *then*

$$t\{x \leftarrow u\}\{y \leftarrow v\} \quad = \quad t\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$$

Find (simple) counter-examples when one hypothesis or the other is not satisfied

Problem Analysis:

Prove that if $x \neq y$ and $x \notin fv(v)$ then $t\{x \leftarrow u\}\{y \leftarrow v\} = t\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$

Two successive substitutions can be done in either order as long as the variables being substituted[1] are different and the first variable doesn't appear free in the second substitution term[2].

Simple Counter Simple Examples:

When <u>$x = y$: $t = x$, $u = a$, $v = b$</u> **gives $x\{x \leftarrow a\}\{x \leftarrow b\} = a$**, however on the other side,

$x\{x \leftarrow b\}\{x \leftarrow b\{x \leftarrow b\}\} = b$

[1]They differ because the order of substitution matters when variable names collide.

a ¡= b.

When <u>$x \in fv(v)$</u>: $t = y$, $u = a$, $v = x$ **gives $y\{x \leftarrow a\}\{y \leftarrow x\} = x$**, however on the other side,

$y\{y \leftarrow x\}\{x \leftarrow a\{y \leftarrow x\}\} = a$

[2]As X is affected in the second $(x \leftarrow a)$ substitution.

Validity Proofs:

To proof the validity of the statement we need to prove the statement in every scenario. Application (t_1t_2), abstraction and induction as in the slides.

Application ~ Case $t = t_1\ t_2$:

Assume the property for t_1 and t_2 => $(t_1\ t_2)\{x \leftarrow u\}\{y \leftarrow v\}$

$= (t_1\{x \leftarrow u\}\ t_2\{x \leftarrow u\})\{y \leftarrow v\}$ by definition

$= t_1\{x \leftarrow u\}\{y \leftarrow v\}\ t_2\{x \leftarrow u\}\{y \leftarrow v\}$ by definition

$= t_1\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}\ t_2\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$ by induction hypothesis

$= (t_1\{y \leftarrow v\}\ t_2\{y \leftarrow v\})\{x \leftarrow u\{y \leftarrow v\}\}$ by definition

$= (t_1\ t_2)\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$ by definition

Which in fact proofs:
$t\{x \leftarrow u\}\{y \leftarrow v\} = t\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$

Abstraction ~ Case $t = \lambda z.t'$:

By Barendregt's convention, assume $z \neq x$, $z \neq y$, $z \notin fv(u)$, $z \notin fv(v)$.

$t\{x \leftarrow u\}\{y \leftarrow v\}$ => $(\lambda z.t')\{x \leftarrow u\}\{y \leftarrow v\}$

$= (\lambda z.t'\{x \leftarrow u\})\{y \leftarrow v\}$ by definition

$= \lambda z.t'\{x \leftarrow u\}\{y \leftarrow v\}$ by definition

$= \lambda z.t'\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$ by induction hypothesis

$= (\lambda z.t'\{y \leftarrow v\})\{x \leftarrow u\{y \leftarrow v\}\}$ by definition

$= (\lambda z.t')\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$ by definition

Which in fact proofs:
$t\{x \leftarrow u\}\{y \leftarrow v\} = t\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$

Induction:

For $t = x$: Left side: $x\{x \leftarrow u\}\{y \leftarrow v\} = u\{y \leftarrow v\}$. Right side: $x\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\}$.

Since $x \neq y$, we have $x\{y \leftarrow v\} = x$. Then the right side is $x\{x \leftarrow u\{y \leftarrow v\}\} = u\{y \leftarrow v\}$.

Both sides coincide.

For $t = y$: Left side: $y\{x \leftarrow u\}\{y \leftarrow v\} = y\{y \leftarrow v\} = v$. Right side: $y\{y \leftarrow v\}\{x \leftarrow u\{y \leftarrow v\}\} = v\{x \leftarrow u\{y \leftarrow v\}\}$.

Now, by the hypothesis $x \notin fv(v)$, the substitution $v\{x \leftarrow w\}$ does not change v for any w (because x does not appear free in v, so there is nothing to substitute). In particular, $v\{x \leftarrow u\{y \leftarrow v\}\} = v$. Therefore both sides equal v. **If t is another variable** z distinct from x and y: Both substitutions leave z unchanged. The statement is valid.