Algorithms for Data Science
Finding Similar Items

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M2 Data Science

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Similar Items










General Problem

Many data mining tasks can be expressed as finding **similar** sets.

- same as **finding near-neighbours in high-dimensional space**

Some applications:

- **similar pages on the Web**: duplicate detection for search engines
- **customer who purchased similar products**
- **images having similar features**

Similarity and Distance

Input – set of high dimensional data points represented as vectors $(x_1\ x_2\ x_3\ \dots\ x_n)$ and a distance function $d(p_i, p_j)$

Problem – find pairs of data points (p_i, p_j) that are close, i.e., in a distance threshold $d(p_i, p_j) \leqslant \tau$

- comparing all pairs would take $\mathcal{O}(N^2)$ (N number of data points) – too expensive
- can be done **much faster**, around $\mathcal{O}(N)$

Documents and Set Similarity

In this lecture, we will study how to find **similar documents** – **near-duplicate pairs**

- plagiarism, mirror pages, articles having the same source

Documents are represented as **sets** / **bags** – we will discuss *how*

Focus on **Jaccard** distance/similarity:

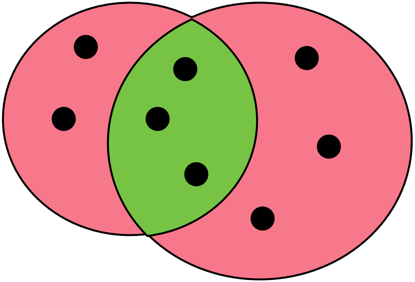
- **Jaccard similarity** of two sets S_1, S_2:

$$\text{sim}(S_1, S_2) = \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|}$$

- **Jaccard distance** is:

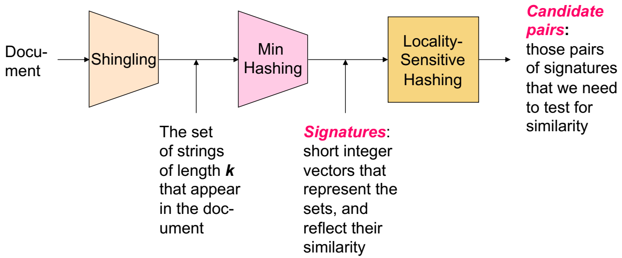
$$d(S_1, S_2) = 1 - \text{sim}(S_1, S2)$$

Jaccard similarity/distance



1. similarity 3/8 – *fraction* of the green area
2. distance 5/8 – *fraction* of the red area

Steps for Finding Similar Documents



1. **shingling** – converting documents to sets
2. **min-hashing** – convert each document to a short signature
3. **locality-sensitive hashing** – reduce the number of pairs of signatures to compare

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Shingling

Naïve way – represent documents as the set of their words – would find many documents that are similar (common words in the language of the document)

- better way – shingling

Shingling: k-shingle = any substring of length k found in the document

- the document is then the set of shingles appearing at least once

Example

- take the document D represented by the string `abcdabd`
- the set of 2-shingles is then $\{\texttt{ab}, \texttt{bc}, \texttt{cd}, \texttt{da}, \texttt{bd}\}$

Shingling in Practice

Principle – k should be picked large enough that the probability of any given shingle appearing in any given document is as low as possible

- assume a document has the 27 chars in the ASCII character set and $k = 5$
- the number of shingles is $27^5 = 14{,}348{,}907$ possible shingles – so $k = 5$ works well for any document that is much smaller than the above size

Shingling in Practice

- in practice, $k = 5$ is good for emails, $k = 10$ is good for large documents
- the size of the sets can be larger that the documents – **hash** the shingles to an integer having a limited number of bits – e.g., for $k = 2$ we only need 10 bits:

$$\{\texttt{ab}, \texttt{bc}, \texttt{cd}, \texttt{da}, \texttt{bd}\} \rightarrow \{342, 825, 312, 54\}$$

- the **similarity/distance** is the Jaccard similarity of sets, applied on the k-shingle sets of each document

Representing Sets of Documents as a Matrix

Conceptually, we will represent the sets of documents as a **Boolean matrix**

- **rows** are the indexes of the possible shingles
- **colums** represent the documents as

Running example

Documents represented as sets $D_1 = \{a, d\}$, $D_2 = \{c\}$, $D_3 = \{b, d, e\}$, $D_4 = \{a, c, d\}$

Table:

Shingle hash	D_1	D_2	D_3	D_4
1	1	0	0	1
2	0	0	1	0
3	0	1	0	1
4	1	0	1	1
5	0	0	1	0

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Why Min-Hashing?

The encoding of sets as boolean values can still be **too costly** – cost = the number of different possible shingles, or the **size of the universal set**

We want to minimize the size of this set, and transform the set into a **signature set**

- in other words, compress the size of the **columns** (=documents) in the matrix

Principle – similarity of signature sets = similarity of shingle sets = similarity of documents

Min-Hashing for Jaccard similarity

Objective – find a hash function h (on the shingle set of the documents), such that:

- if $\text{sim}(D_1, D_2)$ is high, then with high probability, $h(D_1) = h(D_2)$
- if $\text{sim}(D_1, D_2)$ is low, then with high probability, $h(D_1) \neq h(D_2)$

Not all similarity metrics / distances have such a hash function!

- Jaccard has one – Min-Hashing

Min-Hashing for Jaccard similarity

Hash each column C of the table to a small signature $h(C)$:

1. $h(C)$ is small enough to fit in main memory
2. $\mathrm{sim}(C_i, C_j)$ is the same as $\mathrm{sim}(h(C_i), h(C_j))$

Min-Hashing

Shuffle the rows of the matrix using a random permutation π

Define the hash function $h_\pi(C)$ as the first row (in permutation order of π) where we find a value of **1**

Example

Permutation:

π	D_1	D_2	D_3	D_4
2	0	0	**1**	0
5	0	0	1	0
1	**1**	0	0	**1**
4	1	0	1	1
3	0	**1**	0	1

Min-Hash:

D_1	D_2	D_3	D_4
1	3	2	1

Min-Hash Property

D_1	D_4
1	1
0	0
0	1
1	1
0	0

$\text{sim}(D_1, D_4) = 2/3$

Property $\Pr[h_\pi(D_1) = h_\pi(D_2)] = \text{sim}(D_1, D_4)$, for any random permutation π

Proof sketch:

- let $s \in D$ a shingle
- equally likely that $s \in D$ is mapped to the min element; $\Pr[\pi(s) = \min(\pi(D))] = 1/|D|$
- let s be such that $\pi(s) = \min(\pi(D_1 \cup D_4))$
- either $\pi(s) = \min(\pi(D_1))$ if $s \in D_1$, or $\pi(s) = \min(\pi(D_1))$ if $s \in D_4$
- probability that both are true is $\Pr[s \in D_1 \cap D_4]$
- $\Pr[h_\pi(D_1) = h_\pi(D_2)] = \frac{|D_1 \cap D_4|}{|D_1 \cup D_4|} = \text{sim}(D_1, D_4)$.

Min-Hash in Practice

In practice, we need **multiple hash functions**, and thus the similarity of two documents is the fraction of the hash functions in which they agree

- this works because of **the min-hash property**, the similarity of columns is the same as the **expected** similarity of their signatures

Implementation

- permuting rows is **too costly**!
- we can use well-chosen hash functions that achieve a permutation
- the more hash functions we choose, the more exact the computation is – but **more costly**

Signature of a document: $\mathcal{O}(K)$ (number of hash functions)

Min-Hash in Practice

Row	D_1	D_2	D_3	D_4	$h_1 = (2x+1) \mod 5$	$h_2 = (3x+2) \mod 5$
1	1	0	0	1	3	0
2	0	0	1	0	0	3
3	0	1	0	1	2	1
4	1	0	1	1	4	4
5	0	0	1	0	1	2

Algorithm: Choose K permutation functions, initialize $\text{sig}(i, c) = \infty$, and then for each row(=shingle) r:

1. compute $h_1(r), \ldots, h_k(r)$
2. for each column(=document) c: if c has 1, then set $\text{sig}(i, c) = \min(h_i(r), \text{sig}(i, c))$ for $i \in 1, \ldots, K$

Document	Rows with 1s	h_1 values	h_2 values
D_1	{1, 4}	{3, 4} $\rightarrow$ **min = 3**	{0, 4} $\rightarrow$ **min = 0**
D_2	{3}	{2} $\rightarrow$ **min = 2**	{1} $\rightarrow$ **min = 1**
D_3	{2, 4, 5}	{0, 4, 1} $\rightarrow$ **min = 0**	{3, 4, 2} $\rightarrow$ **min = 2**
D_4	{1, 3, 4}	{3, 2, 4} $\rightarrow$ **min = 2**	{0, 1, 4} $\rightarrow$ **min = 0**

	D_1	D_2	D_3	D_4
h_1 signature	3	2	0	2
h_2 signature	0	1	0	0

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Objectives

We achieved smaller documents, but we still need to find a way to compare as few pairs as possible

Idea find a way to only compare pairs that have a similarity above a threshold t

- LSH: use a function $f(x, y)$ that tell whether the pair x, y is a candidate pair for comparison

LSH for Min-Hashing

Assume we have a similarity threshold s

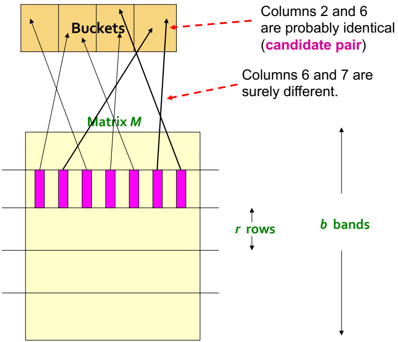
Columns x and y of the signature matrix M are candidate pairs if the signature agrees on at least a fraction s of their rows

- **reminder** we assume that the min-hashed signature output the same expected similarity as the real one

Idea behing LSH for Min-Hashing:

- hash columns of the signature matrix several times, so that only similar columns are likely to hash to the same bucket – candidate pairs are those that hash to the same bucket
- we can divide M into b bands of r rows each

LSH for Min-Hashing



- for each band, hash the portion of the column into *k* buckets
- **candidates** are column pairs (=document pairs) hashing to the same bucket **at least once**
- have to tune *b* and *r* to catch **most** similar pairs, but fewer non-similar pairs

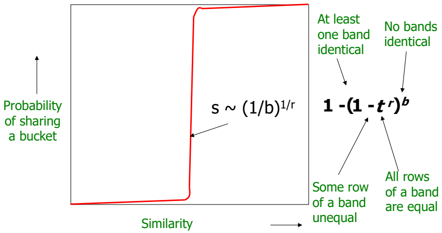
Tuning b and r

Tradeoff number of min-hashes, number of bands b, number of rows per band r

How to compute this?

1. prob. that signatures agree in all rows of one band is s^r
2. prob. that signatures disagree in at least one row is $1 - s^r$
3. the prob. that signatures disagree in at least one row of each of the bands is $(1 - s^r)^b$
4. **candidate pair** if agrees in all the rows of at least one band, prob. is $1 - (1 - s^r)^b$

S-curve for LSH



have to choose the threshold roughly where the probability is $1/2$ – where the curve is steepest

Approximate threshold $t = (1/b)^{1/r}$

Example

Say D_1 and D_2 are 80% similar, $b = 20$, $r = 5$

- prob. D_1,D_2 identical in a given band $0.8^5 = 0.328$
- prob. are not similar in any of the bands: $(1 - 0.328)^{20} = 0.00035$

Only 0.035% of the documents are false negatives (similar but they do not hash in the same bucket anywhere), 99.965% of true positives are found

Example

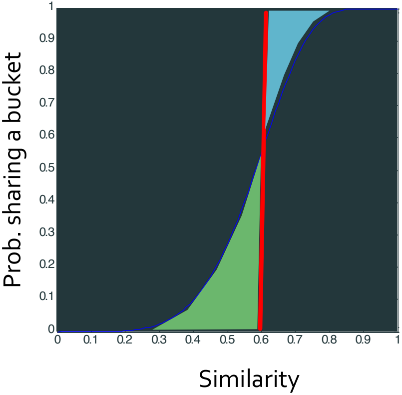
Say D_1 and D_2 are only 30% similar, $b = 20$, $r = 5$

- prob. D_1,D_2 identical in a given band $0.3^5 = 0.00243$
- prob. are similar in at least one of the bands:
  $1 - (1 - 0.00243)^{20} = 0.0474$

Around 4.74% of documents having similarity of 30% end as candidate pairs – false positives (since they are not similar, but we still have to check them)

Using the S-curve

Have to select ***r*** and ***b*** to get the best curve – one which minimizes the **false negatives** (blue) and **false positives** (green)



Putting it All Together

Outline of the steps for similar items:

1. pick a value of k, and construct k-shingles for each documents
2. pick a length n for the min-hash signatures (number of permutations)
3. choose a threshold t, along with b and r such that $br = n$ and $t = (1/b)^{1/r}$
4. construct candidate pairs by applying LSH
5. check each candidate pairs in main memory for similarity

To Go Further

Other **similarity/distance functions** with various application (Sec. 3.5 of [Leskovec et al., 2020])

The mathematical theory behind LSH functions and applying LSH to other similarities (Sec. 3.6 and 3.7 of [Leskovec et al., 2020], [Indyk et al., 1997])

Acknowledgments

The contents and some figures taken from Chapter 3 of [Leskovec et al., 2020]. https://www.mmds.org/

References i

Broder, A. (1997).
On the resemblance and containment of documents.
In *Proceedings of the Compression and Complexity of Sequences (SEQUENCES)*.

Indyk, P., Motwani, R., Raghavan, P., and Vempala, S. (1997).
Locality-preserving hashing in multidimensional spaces.
In *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing (STOC)*, page 618–625.

Leskovec, J., Rajaraman, A., and Ullman, J. (2020).
Mining of Massive Datasets.
Cambridge University Press.